Discrete Structures

Set theory: The set is mathematical concept which is undefined since any try to define the set implies to use the word set say " collection ", " ensemble " sometimes called group, family, system. Although these words are usually reserved for more special types of collection.

Anything in the set is called an " element " it is also undefined concept.

Note: Symbol of the sets usually capital letter say $A, B, X, \dots,$ on the other hand the symbol of elements of a set are small letter say $, b, x, \dots$. If a is an element in the set A we write $a \in A$ and read (belongs to A) and if a is not in A then we write $a \notin A$ and read (a not belongs to A).

Example:

1. The set of the number 130527 is { 1, 3, 0, 5, 2, 7}.
2. The set of the natural numbers N ={ 0, 1, 2, …}.
3. The set of integer numbers Z={ …, -2, -1, 0, 1, 2, …}.

Example:

1. $E=\{x \in N; x = 2n,\ n \in N\}$ even numbers.
2. $O=\{x \in N; x = 2n + 1,\ n \in N\}$ odd numbers.

Definition: The set which is contains no any element called the empty set and is denoted by $\emptyset$.

Example:

$A=\{\ x \in N; 2 < x < 3\} = \emptyset$

Definition: The set A is a subset of the set B and dented by $A \subseteq B$ if each element in A belongs to B.

Example:

$N \subseteq Z \subseteq Q$, where Q is rational number.

Remark: If $A \subseteq B$ and if there exists an element $b \in B$ and $b \notin A$ then we say $A \subset B$. For example, $N \subset Z \subset R$.

Definition: A set A is called equal to a set B if $A \subseteq B$ and $B \subseteq A$, i.e $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Example:

A={2, 4, 6}, $B=\{x; x = 2n, n = 1, 2, 3\}$, C={ 2, 6, 4} then A= B= C.

Theorem: Let A, B, C are sets, then

1. $A \subseteq A$.
2. If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$.
3. If $A \subset B$, $B \subset C$ then $A \subset C$.

Definition: (the Universal set)

All set which deal with are subsets from " Big " set or another set then this set is called universal set.

Example:

A={1, 2, 5}, B={2, 4, 5}, C={2, 9, 10}, so that U={1, 2, 3, 4, 5, 6, 9,10}.

Definition: (Power set)

Let A be any set, the set of all subsets of the set A is called the power set and it is denoted by $P(A)$ or 2^A, i.e $P(A)$ =$\{B; B \subseteq A\}, B \in P(A)$ iff $B \subseteq A$.

Example:

1. A={1, 2, 3}, find $P(A)$?
   $P(A) = \{\emptyset, \{1, 2, 3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2, 3\}\}$.

2. $A = \{a, b\}$, then $P(A) = \{\emptyset, \{a, b\}, \{a\}, \{b\}\}$.

Operation on sets

Definition: (the intersection of set)

The intersection of a set A and a set B is the following set $A \cap B = \{x, x \in A \text{ and } x \in B\}$.

Remark: Let A, B are two sets, then

1. $A \cap B \subseteq A, A \cap B \subseteq B$.
2. $A \subseteq B$ iff $A \cap B = A$.

Definition: (the union of set)

The union of a set A and B is the set

$A \cup B = \{x, x \in A \text{ or } x \in B\}$.

Example:

1. Let $A = \{a, b, c, d\}$, $B = \{f, b, d, g\}$ then $A \cap B = \{b, d\}$,

   $A \cup B = \{a, b, c, d, f, g\}$.

2. Let $E =$ even numbers and $O =$ odd numbers then $E \cap O = \emptyset$, $E \cup O = N$.

Proposition: Let $A, B, and\ C$ are sets then

1. $A \cup A = A$, $A \cap A = A$.
2. $A \cap B = B \cap A$, $A \cup B = B \cup A$ commutative.
3. $A \cup (B \cup C) = (A \cup B) \cup C$ associative.

   $A \cap (B \cap C) = (A \cap B) \cap C$ associative.

Definition: (the Disjoin set)

Let A, B are two sets, then A, B is called disjoin set if $A \cap B = \emptyset$.

Example: E and O are disjoin set since $E \cap O = \emptyset$.

Definition: (the Difference)

The difference between the sets A, B is the set $A - B = \{x, x \in A\ and\ x \notin B\}$.

Example:

Let $A = \{a, b, c, d\}$, $B = \{f, b, d, g\}$ then $A - B = \{a, c\}$.

Definition: (Complement)

The complement of the set A is the set which contains elements not belong to A, and denoted by A^{C}. i.e. $A^{C} = \{x, x \notin A\} = U - A$.

Example:

Let $E =$even numbers and $U = N$ then $E^{C} = U - E = O$ odd numbers.

Distributive Laws:

Let $A, B, and\ C$ are sets, then

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof:

1) Let $x \in A \cap (B \cup C) \rightarrow x \in A\ and\ x \in (B \cup C) \rightarrow x \in A and\ (x \in B\ or\ x \in C) \rightarrow$

$(x \in A\ and\ x \in B) or\ (x \in A\ and\ x \in C) \rightarrow x \in (A \cap B) or\ x \in (A \cap C) \rightarrow$

$x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ …..(1)

Now,

Let $y \in (A \cap B) \cup (A \cap C) \rightarrow y \in (A \cap B)\ or\ y \in (A \cap C) \rightarrow$

$(y \in A\ and\ y \in B)\ or\ (y \in A\ and\ y \in C) \rightarrow$

$y \in A\ and\ (y \in B\ or\ y \in C) \rightarrow y \in A\ and\ y \in (B \cup C) \rightarrow$

$y \in A \cap (B \cup C)$

$\therefore (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$…..(2)

From (1) and (2) we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

De Morgan's Law:

Let A, B be any two sets, then

1. $(A \cup B)^C = A^C \cap B^C$
2. $(A \cap B)^C = A^C \cup B^C$

Proof: 1) Let $x \in (A \cup B)^{C} \to x \notin A \cup B \to x \notin A \wedge x \notin B \to x \in A^{C} \wedge x \in B^{C} \to$

$x \in A^{C} \cap B^{C}$

$\therefore (A \cup B)^{C} \subseteq A^{C} \cap B^{C}$……(1)

Let $y \in A^{C} \cap B^{C} \to y \in A^{C} \wedge y \in B^{C} \to y \notin A \wedge y \notin B \to y \notin A \cup B \to$

$y \in (A \cup B)^{C}$

$\therefore A^{C} \cap B^{C} \subseteq (A \cup B)^{C}$…….(2)

From (1) and (2) we get $(A \cup B)^{C} = A^{C} \cap B^{C}$

Mathematical Induction

Suppose the statement to be proved can be put in the form $\forall n \geq n_0\ P(n)$, where n_0 is some fixed integer. That is, suppose we wish to show that $P(n)$ is true for all integers $n \geq n_0$. The following result shows how this can be done. Suppose that (a) $P(n_0)$ is true and (b) If $P(k)$ is true for some $k \geq n_0$, then $P(k+1)$ must also be true. Then $P(n)$ is true for all $n \geq n_0$. This result is called the ***principle of mathematical induction***.

Example:

Show, by mathematical induction, that for all $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution: Let $P(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

1. Since $\frac{1(1+1)}{2} = \frac{2}{2} = 1$, hence $P(1)$ is true

2. Suppose that $P(k)$ is true i.e. $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$

We have to show that $1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$?

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$$1+2+3+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}=\frac{(k+1)[k+1+1]}{2}=\frac{(k+1)[(k+1)+1]}{2}$$

$1+2+3+\cdots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}$ so that, $P(k+1)$ is true . Hence $P(n)$ is true $\forall\, n \geq 1$

Example:

Prove that $\sum_{k=1}^{n}(2k-1)^2=\frac{n(2n-1)(2n+1)}{3}\ \ \forall\, n \in N$

Solution: Let $P(n)=\sum_{k=1}^{n}(2k-1)^2=1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$

Since $1^2=\frac{1(2-1)(2+1)}{3}=1$, hence $P(1)$ is true.

Suppose that $P(k)$ is true, i.e. $1^2+3^2+5^2+\cdots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$ …(*)

Now, we prove that $P(k+1)$ is true

i.e. $1^2+3^2+5^2+\cdots+(2k-1)^2+(2(k+1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$?

$$\begin{aligned}1^2+3^2+5^2+\cdots+(2k-1)^2+(2(k+1)-1)^2 &= \frac{k(2k-1)(2k+1)}{3}+(2k+1)^2\\ &= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}\\ &= \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}\\ &= \frac{(2k+1)[2k^2+5k+3]}{3}\\ &= \frac{(2k+1)[(k+1)(2k+3)]}{3}\\ &= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}\end{aligned}$$

So that, $P(k+1)$ is true.

Problem:

Prove that 1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2 \ \forall \ n \in N$

2. $3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3}{2}(3^n - 1)$

Relation and Functions

Example:

1. " x *is less than* y "
2. " x *divides* y "
3. " x *is the wife of* y "
4. " *the square of* x *plus the square of* y *is sixteen* ", i.e. " $x^2 + y^2 = 16$ ".

Ordered Pairs

An ordered pair consists of two elements, say a and b, in which one of them, say a as the first element and the other as the second element. An ordered pair is denoted by (a, b).

Remark:

1. An ordered pair (a, b) can be defined by $(a, b) = \{\{a\}, \{a, b\}\}$.
2. Ordered pairs can have the same first and second elements such as $(1,1), (4,4)$ and (5,5).
3. If $a = b$ then $(a, b) = \{\{a\}\}$.
4. If $a \neq b$ then $(a, b) \neq (b, a)$.

Theorem: Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Product Set: Let A and B be two sets. The product set of A and B consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$. It is denoted by $A \times B$ which reads "A *cross* B". i.e. $A \times B = \{(a, b), a \in A, b \in B\}$.

Example:

1. $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then the product set $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

2. Let $w = \{s, t\}$. Then $w.w = \{(s, s), (s, t), (t, s), (t, t)\}$.

Remark:

1. If set A has n elements and set B has m elements then the product set $A \times B$ has $n.m$ elements.

2. In general $A \times B \neq B \times A$.

3. If A is any set and $B = \emptyset$ then $A \times B = \emptyset$.

4. If $A \neq \emptyset$ and $B \neq \emptyset$ then $A \times B = B \times A$ iff $A = B$.

Relations:

Let A and B are two sets, every subset of $A \times B$ is called relation from A to B, denoted by R.

Remark:

1. We say R relation on A if $R \subseteq A \times A$.

2. If $(a, b) \in R$, then denoted by $a\ R\ b$.

3. If $(a, b) \notin R$, then denoted by $a \not R\ b$.

Example:

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then $R = \{(1, a), (1, b), (3, a)\}$ is a relation from A to B. Furthermore, $1\ R\ a, 2 \not R\ b, 3\ Ra, 3 \not R\ b$.

2. Let $w = \{a, b, c\}$. then $R = \{(a, b), (a, c), (c, c), (c, b)\}$ is a relation in w. Moreover, $a \not R\ a,\ b \not R\ a,\ c\ R\ c,\ a\ R\ b$.

Reflexive relation: Let R be a relation in a set A, i.e. Let R be a subset of $A \times A$. then R is called a reflexive relation if for every $a \in A$, $(a, a) \in R$. In other words, R is reflexive if every element in A is related to its.

Example:

1. Let $V = \{1, 2, 3, 4\}$ and $R = \{(1,1), (2,4), (3,3), (4,1), (4,4)\}$. then R is not reflexive since $(2, 2)$ dose not belong to R. Notice that all ordered pairs (a, a) must belong to R in order for R to be reflexive.
2. Let R be the relation in the real numbers defined by the open sentence " $x < y$". Then R is not reflexive since $a \nless a$ for every real number a.
3. Let Q be a family of sets, and let R be the relation in Q defined by " $x\ is\ a\ subset\ of\ y$ ". Then R is a reflexive relation since every set is a subset of itself.

Symmetric relation: Let R be a relation in a set A. Then R is called a symmetric relation if $(a, b) \in R$ implies $(b, a) \in R$ that is, if a related to b, then b is also related to a.

Example:

1. Let $S = \{1, 2, 3\}$ and let $R = \{(1,3), (4,2), (2,4), (2,3), (3,1)\}$. Then R is not a symmetric relation since $(2, 3) \in R$ but $(3, 2) \notin R$.
2. Let R be the relation in the natural numbers N which is defined by " $x\ divides\ y$". Then R is not symmetric since $2\ divides\ 4$ but $4\ does\ not\ divides\ 2$. i.e. $(2, 4) \in R$ but $(4, 2) \notin R$.

Transitive relation: A relation R in a set A is called a transitive relation if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

Example:

1. Let R be the relation in the real numbers defined by "$x < y$ ". Then if $a < b$ and $b < c$ implies $a < c$. Thus R is a transitive relation.

2. Let $w = \{a, b, c\}$, and let $R = \{(a, b), (c, b), (b, a), (a, c)\}$. Then R is not a transitive relation since $(c, b) \in R$ and $(b, a) \in R$ but $(c, a) \notin R$.

Equivalence relation: A relation R in a set A is an equivalence relation if

1. R is reflexive, that is, for every $a \in A$, $(a, a) \in R$.
2. R is symmetric, that is, $(a, b) \in R$ implies $(b, a) \in R$.
3. R is transitive, that is, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

Example: Let $A = \{1, 2, 3\}$. Determine whether the relation is reflexive, symmetric or transitive.

1. $R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$.
2. $R = \{(1,3), (1,1), (3,1), (1,2), (3,3)\}$.
3. $R = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3, 2)\}$.
4. $R = \{(1,2), (1,3), (2,3), (1,1), (3,1), (3,3), (2,1), (2,2)\}$.
5. $R = \{(1,3), (3,1), (2,2)\}$.

Solution:

1. Reflexive, symmetric, transitive.
2. Not reflexive, not symmetric, not transitive.
3. Not reflexive, not symmetric, transitive.
4. Reflexive, not symmetric, not transitive.
5. Not reflexive, symmetric, not transitive.

Example:

Consider the relation R that is define on Z by $a\ R\ b \leftrightarrow \exists\ k \in Z,\ a - b = 3k$.

Solution:

$(3, 0) \in R$ since $3 - 0 = 3 = 3(1), k = 1$

$(0, 3) \in$ R since $0 - 3 = -3 = 3(-1),\ k = -1$

$(15, 9) \in R$ since $15 - 9 = 6 = 3(2),\ k = 2$

$(-1, -2) \notin R$ since $-1 - (-2) = 1 = 3k, k = \frac{1}{3} \notin Z$

Now,

1. R is reflexive?

Let $a \in Z\ \rightarrow a - 0 = 3(0), k = 0 \rightarrow a\ R\ a$. So that, R is reflexive.

2. R is symmetric?

Let $a\ R\ b$, we have to show $b\ R\ a$?

Since $a\ R\ b \rightarrow \exists\ k \in Z; a - b = 3k$

Now,

$b - a = -(a - b) = -3k = 3(-k) = 3k_1$ s.t. $k_1 = -k$

$\therefore b - a = 3k_1 \rightarrow b\ R\ a$ so that R is symmetric.

3. R is transitive?

Let $a\ R\ b\ \wedge\ b\ R\ c$, we have to show $a\ R\ c$?

$a - b = 3k_1\ \wedge\ b - c = 3k_2;\ \ k_1, k_2 \in Z$

$a - c = a - b + b - c = 3k_1 + 3k_2 = 3(k_1 + k_2) = 3k;\ where\ k = k_1 + k_2$

$\therefore a\ R\ c$

So that from (1), (2), and (3) R is equivalence relation.

Problem:

Let $A = \{1, 2, 3, 4, 5\}$ and R is defined on A by $R = \{(x, y),\ x + y = 5\}$. Is R an equivalence relation.

Example:

Let $A = Z \times Z - \{0\}$, we define the relation R on A by $(a, b)\ R\ (c, d) \leftrightarrow ad = bc$.

Solution:

$(0,1)\ R\ (0,2)$

$(2,4)\ R\ (10,20)$

$(5,15)\ \not R\ (25,10)$

1. R is reflexive relation?

   Let $(a, b) \in A$, it is clear that $(a, b)\ R\ (a, b)$ since $ab = ba$.

2. R is symmetric relation?

   Let $(a, b)\ R\ (c, d) \to ad = bc \to bc = ad \to cb = da \to (c, d)\ R\ (a, b)$.

3. R is transitive relation?

   Let $(a, b)\ R\ (c, d) \quad \wedge \quad (c, d)\ R\ (e, f)$, we have to show that $(a, b)\ R\ (e, f)$?

   Since $ad = bc \ \ \wedge \ \ cf = de \to c = \frac{ad}{b} \to cf = \frac{ad}{b} f = de \to adf = bde \to$

   $af = be$, $\therefore (a, b)\ R\ (e, f)$ so that by (1), (2), and (3) R is equivalence relation.

Functions

Definition: Let f be a relation defined from a set A to set B, then f is called a function " $denoted\ by\ f: A \to B$ " if the following holds:

1. $\forall a \in A, \exists\ b \in B$ such that $(a, b) \in f$.
2. If $(a, b_1) \in f\ and\ (a, b_2) \in f \quad then \quad b_1 = b_2$.

The set A is called the domain of the function f, and B is called the co-domain or range of f. Further, if $a \in A$ then the element in B which is assigned to a is called image of a and is denoted by $f(a)$.

Example:

1. Let $A = \{a, b, c, d\}$ and $B = \{a, b, c\}$. Define a function f of A into B be the correspondence $f(a) = b,\ f(b) = c,\ f(c) = c\ and\ f(d) = b$. By this definition, the image for example are $b\ and\ c$.
2. Let $A = \{1, 2, 3\}$, $g: A \to A$ defined by $g = \{(1, 2), (3, 1)\}$ is not function because not for each $a \in A$, i.e. $2 \in A, \nexists b \in\ s.t\ (2, b) \in g$.
3. $h: A \to A$ define by $h: \{(1, 3), (2, 1), (1, 2), (3, 1)\}$ is not function since $(1, 2) \in h$ and $(1, 3) \in h$ but $2 \neq 3$.

Example: Let $A = \{m, n, o, p\}$ and $B = \{1, 2\}$. Determine whether the relation R from $A\ to\ B$ is a function. If it is function, give its range.

1. $R = \{(m, 1), (n, 1), (o, 1), (p, 1)\}$.
2. $R = \{(m, 1), (n, 2), (m, 2), (o, 1), (p, 2)\}$.
3. $R = \{(m, 2), (p, 1), (n, 1), (o, 1), (m, 1)\}$.

Solution:

1. Yes function, $Range(R) = \{1\}$.
2. No.
3. No.

Problem: Let $A = \{w, x, y, z\}$ and $B = \{1, 2\}$. Determine whether the relation R from $A\ to\ B$ is a function. If it is function, give its range.

1. $R = \{(w, 2), (x, 2), (y, 2), (z, 2)\}$.
2. $R = \{(w, 1), (x, 2), (w, 2), (y, 1), (z, 2)\}$.
3. $R = \{(w, 2), (z, 1), (x, 1), (y, 1), (w, 1)\}$.

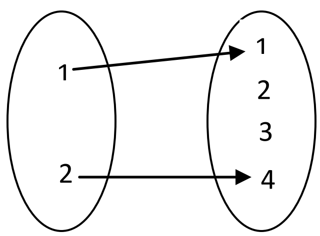
Equal function

If $f\ and\ g$ are two functions defined on the same domain D and if $f(a) = g(a)$ for every $a \in D$, then the functions $f\ and\ g$ are equal and we write $f = g$.

Example:

Let $f(x) = x^2$ where x is a real number. Let $g = x^2$ where x is a complex number, then the function f is not equal to g since they have different domains.

Example: Let the function f be defined by the diagram



Let a function g be defined by the formula $g(x) = x^2$ where the domain of g is the set $\{1, 2\}$, then $f = g$ since they both have the same domain and since $f\ and\ g$ assign the same image to each element in the domain.

Range of a function

Let f be a mapping of $A\ into\ B$, that is, $f: A \to B$ we define the range of f to consist of those elements in B which appear as the image of at least one element in A. We denote the range of $f: A \to B$ by $f(A)$; i.e. $f(A) = \{b \in B; (a, b) \in f\}$.

Identity function

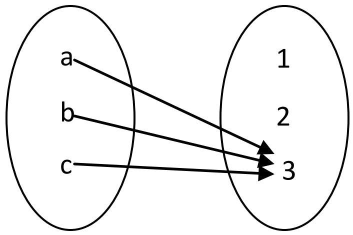
Let A be any set. Let the function $f: A \to A$ be defined by the formula $f(x) = x$ for each element in A then f is called the identity function and denote by I or I_A.

Constant function

A function f of $A\ into\ B$ is called a constant function if the same element $b \in B$ is assigned to every element in A. In other words, $f: A \to B$ is a constant function if the range of f consists of only one element.

Example:

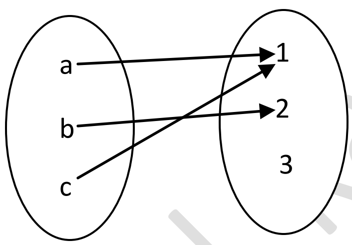
Let the function f be defined by the diagram



Then f is a constant function since 3 is assigned to every element in A.

Example:

Let the function be defined by the diagram



Then f is not a constant function since the range of f consists of both 1 and 2.

Definition: Let f be a function of A into B. Then f is called

1. One – one [injective]

   If $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$

   or

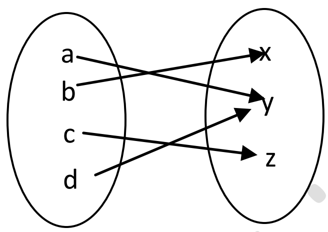
   If $f(x_1) = f(x_2) \rightarrow x_1 = x_2$
2. Onto [surjective]

   If for each $b \in B$, there exists $a \in A$ such that $f(a) = b$. i.e. $f(A) = B$.
3. Bijective

   If f is both injective and surjective.

Example:

1. The identity function is injective since $I(x_1) = I(x_2) \rightarrow x_1 = x_2$.
2. The identity function is surjective since for each $b \in B$, $\exists\, a \in A$ such that $f(A) = B$.
3. Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$. Define by $f(a) = b, f(b) = c, f(c) = c$ and $f(d) = b$ then $f(A) = \{b, c\} \neq B$ so that f is not onto.
4. Let $= \{a, b, c, d\}, B = \{x, y, z\}$, Let $f: A \rightarrow B$ define by



f is not injective because $f(a) = f(d)$ but $a \neq d$. f is surjective, notice that $f(A) = \{x, y, z\} = B$, thus f is onto.

Problem:

Let $A\ and\ B$ are two sets and the function from A to B are given. Determine whether the function is one to one or onto (or both or neither).

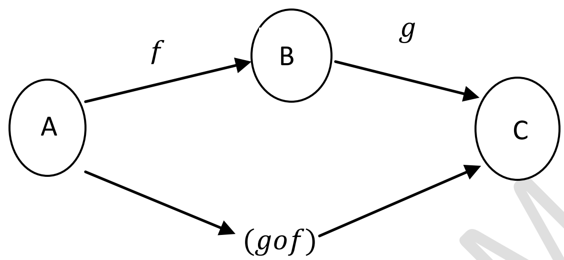
1. $A = \{0.2, 0.07, 5\}, \quad B = \{r, s, t\}$,

   $f = \{(5, t), (0.2, s), (0.07, r)\}$
2. $A = \{m, n, o\}, \quad B = \{2, 1, 5\}$,

   $f = \{(m, 5), (o, 1), (n, 1)\}$

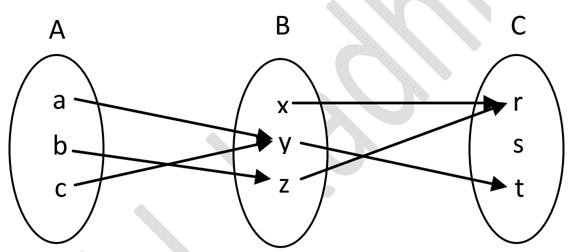
Composition function

Let $f: A \to B$ and $g: B \to C$ be two functions then we define a function $gof: A \to C$ by $(gof)(a) = g(f(a))$ $\forall a \in A$. This new function is called the composition function of f and g.



Example:

Let $f: A \to B$ and $g: B \to C$ be defined by the diagram



We compute $(gof): A \to C$ by its definition

$$(gof)(a) = g(f(a)) = g(y) = t$$

$$(gof)(b) = g(f(b)) = g(z) = r$$

$$(gof)(c) = g(f(c)) = g(y) = t$$

Example:

Let $f: N \to N,$ $g: N \to N$ such that $f(x) = 3x^2$ and $g(x) = 4x^2 + 2$. Now,

$(gof)(x) = g(f(x)) = g(3x^2) = 4(3x^2)^2 + 2 = 36x^4 + 2.$

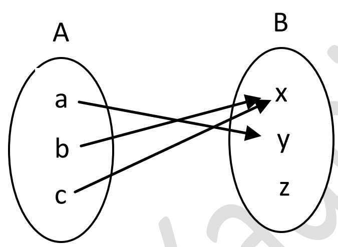
$(fog)(x) = f(g(x)) = f(4x^2+2) = 3(4x^2+2)^2 = 3(16x^4+16x^2+4)$. It is clear that $gof \neq fog$.

Inverse of function

Let f be a function of A into B, and let $b \in B$. Then the inverse of b, denoted by $f^{-1}(b)$ consists of those elements in A which are mapped onto b, that is, those elements in A which have be as their image. i.e. if $f: A \to B$ then $f^{-1}(b) = \{x; x \in A, f(x) = b\}$. Notice that $f^{-1}(b)$ is always a subset of A.

Example:

Let the function $f: A \to B$ be defined by the diagram



Then $f^{-1}(x) = \{b, c\}$, $f^{-1}(y) = \{a\}$, $f^{-1}(z) = \emptyset$.

Statements

Statements is a verbal sentence helpful will be denoted by the letters $, q, r, \ldots\ldots$. The fundamental property of a statement is that it is either true or false but not both. The truth fullness or falsity of a statement is called its truth value. Some statements are composite, that is, composed of sub statements are various connectives which will be discussed sub sequently.

Example:

1. " $Hiba\ is\ a\ nice\ and\ a\ clever\ girl$ " is a composite statement with sub statements " $Hiba\ is\ a\ nice$ " and " $Hiba\ is\ a\ clever\ girl$ " .
2. " $Where\ are\ you\ going?$ " is not a statement since it is neither true nor false.

3. $"John\ is\ sick\ or\ old"$ is a composite statement with sub statements $"John\ is\ sick"$ or $"John\ is\ old"$. A fundamental property of a composite statement is that, its truth value is completely determined by the truth value of its sub statements and the way they are connected to form the composite statement.

Conjunction

Any two statements can be combined by the word $"and"$ to form a composite statement which is called the conjunction of the original statements. The conjunction of the two statements $p\ and\ q$ is denoted by $p \wedge q$.

Example:

Let p be $"It\ is\ rainig"$ and let q be $"The\ sun\ is\ shining"$. Then $p \wedge q$ denotes the statement $"It\ is\ rainig\ and\ the\ sun\ is\ shining"$. The truth value of the composite statement $p \wedge q$ satisfies the following property:

If p is true and q is true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false. In other words, the conjunction of two statement is true only if each component is true.

Example:

Consider the following four statements

1. Paris is in France and $2 + 2 = 5$.
2. Paris is in England and $2 + 2 = 4$.
3. Paris is in England and $2 + 2 = 5$.
4. Paris is in France and $2 + 2 = 4$.

It is clear that only (4) is true. Each of the other statements is false since at least one of its components is false.

Truth table of " $p \wedge q$ " can be written in the form

p	q	$p \wedge q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***F***

Disjunction

Any two statements can be combined by the word " or " to form a new statement which is called the disjunction of the original two statements. The disjunction of statements $p\ and\ q$ is denoted by $p \vee q$.

Example:

Let p be " $He\ studied\ French\ at\ the\ university$ ", and let q be " $He\ lived\ in\ France$ " then $p \vee q$ is the statement " $He\ studied\ French\ at\ the\ university\ or\ he\ lived\ in\ France$ ".

The truth value of the composite statement $p \vee q$ satisfies the following property:

If p is true or q is true or both $p\ and\ q$ are true, then $p \vee q$ is true, otherwise, $p \vee q$ is false. In other words, the disjunction of two statements is false only if $p\ and\ q$ are false. The truth table of " $p \vee q$ " can be written in the form

p	q	$p \vee q$
T	***T***	***T***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Example:

Consider the following four statements

1. Paris is in France or $2 + 2 = 5$.
2. Paris is in England or $2 + 2 = 4$.
3. Paris is in France or $2 + 2 = 4$.
4. Paris is in England or $2 + 2 = 5$.

It is clear that only (4) is false. Each of the other statements is true since at least one of its components is true.

Negation

Given any statement p, another statement, called the negation of p, can be formed by writing "$It\ is\ false\ that$ ..." before p or, if possible by inserting in p the word "not". The negation of p is denoted by $\sim p$.

Example:

Consider the following three statements

1. Paris is in France.
2. It is false that Paris is in France.
3. Paris is not in France.

Then (2) and (3) are each the negation of (1).

Example:

Consider the following statements

1. $2 + 2 = 5$
2. It is false that $2 + 2 = 5$
3. $2 + 2 \neq 5$

Then (2) and (3) are each the negation of (1).

The truth value of the negation of a statement satisfies the following property:

If p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.

p	$\sim p$
T	*F*
F	*T*

Conditional

Many statements, especially in mathematics are of the form " $if\ p\ then\ q$ " such statements are called conditional statements and are denoted by $p \to q$. The truth value of the conditional statement $p \to q$ satisfies the following property:

The conditional $p \to q$ is true unless p is true and q is false.

The truth table of " $p \to q$ " can be written in the form

p	q	$p \to q$
T	*T*	*T*
T	*F*	*F*
F	*T*	*T*
F	*F*	*T*

Remark:

Consider the conditional proposition $p \to q$ and other simple conditional proposition which contain p and q, i.e. $p \to q,\ \sim p \to \sim q\ and\ \sim q \to \sim p$, called respectively, the converse, inverse, and contra positive propositions.

The truth table of these four propositions are as follows:

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	*T*	*F*	*F*	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*	*T*	*T*	*F*
F	*T*	*T*	*F*	*T*	*F*	*F*	*T*
F	*F*	*T*	*T*	*T*	*T*	*T*	*T*

Example:

Let p: $Noor\ at\ home.$

q: $Noor\ answer\ to\ the\ phone.$

$p \rightarrow q$: $If\ Noor\ at\ home\ then\ she\ will\ answer\ to\ the\ phone.$

$q \rightarrow p$: $If\ Noor\ answer\ to\ the\ phone\ then\ she\ is\ at\ home.$

$\sim p \rightarrow \sim q$: $If\ Noor\ is\ not\ at\ home\ then\ she\ is\ not\ answer\ to\ the\ phone.$

$\sim q \rightarrow \sim p$: $If\ Noor\ is\ not\ answer\ to\ the\ phone\ then\ she\ is\ not\ at\ home.$

Biconditional

Another common statement is of the form " $p\ if\ and\ only\ if\ q$ " or simply, " $p\ iff\ q$ ". Such statements are called biconditional statements and denoted by $p \leftrightarrow q$.

The truth value of the biconditional statement $p \leftrightarrow q$ satisfies property:

If $p\ and\ q$ have the same truth value, then " $p \leftrightarrow q$ " is true,

If $p\ and\ q$ have opposite truth value, then " $p \leftrightarrow q$ " is false.

Example:

Consider the following statements

1. Paris is in France iff $2 + 2 = 5$.
2. Paris is in England iff $2 + 2 = 4$.
3. Paris is in France iff $2 + 2 = 4$.
4. Paris is in England iff $2 + 2 = 5$.

According , (3) and (4) are true while (1) and (2) are false.

The truth table written as follows

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Equivalence

Two statements are said to be logically equivalent if their truth table are identical. We denote the logical equivalent of $p\ and\ q$ by " $\equiv$ ".

Example:

The truth tables of $(p \to q) \wedge (q \to p)\ \ and\ \ p \leftrightarrow q$ are as follows:

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Hence, $(p \to q) \wedge (q \to p)\ \equiv\ p \leftrightarrow q$

Example:

The truth tables below show that $p \rightarrow q$ *and* $\sim p \vee q$ are logically equivalent, i.e. $p \rightarrow q \equiv \sim p \vee q$

p	q	$p \rightarrow q$
T	*T*	*T*
T	*F*	*F*
F	*T*	*T*
F	*F*	*T*

p	q	$\sim p$	$\sim p \vee q$
T	*T*	*F*	*T*
T	*F*	*F*	*F*
F	*T*	*T*	*T*
F	*F*	*T*	*T*

Problems

Show that

1. $\sim(p \wedge q) \equiv \sim p \wedge \sim q$
2. $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Some of questions and solutions of discrete structures

1. Prove that the statement is true for every positive integer n.

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

a. p_1 is true, since $\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{2(1)+1}\rightarrow\frac{1}{(1)(3)}=\frac{1}{3}\rightarrow\frac{1}{3}=\frac{1}{3}$

b. assume that p_k is true

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Hence,

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=$$

$$\frac{k}{2k+1}+\frac{1}{(2k+2-1)(2k+2+1)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=$$

$$\frac{k(2k+3)+1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=$$

$\frac{(k+1)}{2(k+1)+1}$ thus, p_{k+1} is true.

2. Prove that the statement is true for every positive integer n.

$$2+7+12+\cdots+(5n-3)=\frac{1}{2}n(5n-1)$$

a. p_1 is true, since $5(1)-3=\frac{1}{2}(1)(5(1)-1)\rightarrow 2=\frac{1}{3}(4)\rightarrow$

$$2=2$$

b. assume that p_k is true

$$2+7+12+\cdots+(5k-3)=\frac{1}{2}k(5k-1)$$

Hence,

$$2+7+12+\cdots+(5k-3)+(5(k+1)-3)=$$

$$\frac{1}{2}k(5k-1)+(5(k+1)-3)= \frac{1}{2}k(5k-1)+5k+5-3=$$

$$\frac{5}{2}k^2-\frac{1}{2}k+5k+2=\frac{5k^2-k+10k+4}{2}=\frac{1}{2}[5k^2+9k+4]=$$

$$\frac{1}{2}[(k+1)(5k+4)]=\frac{1}{2}[(k+1)(5(k+1)-1)]$$

thus, p_{k+1} is true.

3. Prove that the statement is true for every positive integer n.

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

a. p_1 is true, since $1^2=\frac{1(1+1)(2+1)}{6}\to 1=\frac{6}{6}\to 1=1$

b. assume that p_k is true

$$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$$

Hence,

$$1^2+2^2+3^2+\cdots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$\frac{k(k+1)(2k+1)+6(k+1)^2}{6}=(k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]=$$

$$(k+1)\left[\frac{2k^2+7k+6}{6}\right]=(k+1)\left[\frac{(k+2)(2k+3)}{6}\right]=$$

$\frac{(k+1)\left((k+1)+1\right)(2(k+1)+1)}{6}$ thus, p_{k+1} is true.

4. Let $A=\{1,2,3,4\}$. Determine whether the relation is reflexive, symmetric or transitive.

a. $R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$

b. $R=\{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$

c. $R=\{(1,2),(1,3),(3,1),(1,1),(3,3),(3,2),(1,4),(4,2),(3,4)\}$

solution

a. Reflexive, symmetric, transitive.

b. Not reflexive, not symmetric, not transitive.

c. Not reflexive, not symmetric, transitive.

٥. Determine whether the relation R on the set A is an equivalence relation:

a. $A = \{a, b, c, d\}$, $R = \{(a, a), (b, a), (b, b), (c, c), (d, d), (d, c)\}$

b. $A = \{1, 2, 3, 4\}$,

$R = \{(1, 1), (1,2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$

Solution

a. 1) R is reflexive 2) R is not symmetric 3) R is transitive

hence, R is not an equivalence relation since R is not symmetric.

b. 1) R is reflexive 2) R is not symmetric 3) R is not transitive

hence, R is not an equivalence relation since R is not symmetric and not transitive.

6. Let $A = Z$, the set of integers, and let R be defined by $a\ R\ b$ if and only if $a \le b$. Is R an equivalence relation?

Solution

1) Since $\forall\ a \in A,\ a \le a$ thus, R is reflexive.

2) If $a \le b, it\ need\ not\ \ b \le a\ \ so\ that\ R\ is\ not\ symmetric.$

3) R is transitive, since $a \le b\ and\ b \le c\ imply\ that\ a \le c.$

We see that R is not an equivalence relation.

7. Let $A = R \times R$. Define the following relation R on A: $(a, b)R(c, d)$ iff $a^2 + b^2 = c^2 + d^2$. Show that R is an equivalence relation.

solution

1) R is reflexive, since $\forall(a, b) \in A$, $(a, b)R(a, b)$

   Since $a^2 + b^2 = a^2 + b^2$

2) R is symmetric, since $(a, b)R(c, d)$ that is, $a^2 + b^2 = c^2 + d^2$ implies

   $c^2 + d^2 = a^2 + b^2$ hence, $(c, d)R(a, b)$.

3) R is transitive, since $(a, b)R(c, d)$ that is $a^2 + b^2 = c^2 + d^2$ and $(c, d)R(e, f)$ that is $c^2 + d^2 = e^2 + f^2$ implies

   $a^2 + b^2 = c^2 + d^2 = e^2 + f^2$ hence, $a^2 + b^2 = e^2 + f^2$ therefore, $(a, b)R(e, f)$

8. Verify that the formula yields a function from $A\ to\ B$.

   $A = B = Z; \quad f(a) = a^2$

   Solution

   Each integer has a unique square that is also an integer